

Lattice Codes for the Binary Deletion Channel

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Abstract

The construction of deletion codes for the Levenshtein metric is reduced to the construction of codes over the integers for the Manhattan metric by run length coding. The latter codes are constructed by expurgation of translates of lattices. These lattices, in turn, are obtained from Construction A applied to binary codes and \mathbb{Z}_4 -codes. A lower bound on the size of our codes for the Manhattan distance are obtained through generalized theta series of the corresponding lattices.

Keywords: Deletion codes, lattice, Lee metric, Construction A, weight enumerator, ν -series

I. INTRODUCTION

Coding for the binary deletion channel remains a major challenge for coding theorists. Part of the reason for this is that the use of standard block algebraic coding techniques (parity-checks, cosets, syndromes) is precluded due to the specificity of the channel which produces output vectors of variable lengths. A variation of this channel is the so-called segmented deletion channel where at most a fixed number of errors can occur within segments of given size [17], [16]. Because of this restriction, the segmented deletion channel does not alterate the number of runlengths if they are long enough. Hence, if we view the channel in terms of input/output runlengths, the input and output vectors have the same dimension (assuming long enough runlengths). In this case, algebraic coding techniques can be used.

In this paper, we construct lattice-based codes, which, in principle, can be decoded when obtained via Construction A from Lee metric codes with known decoding algorithms [6]. The proposed code constructions are analogous to the so-called (d, k) -codes in magnetic recording where each codeword contains runs of zeros of length at least d and at most k while each run of ones has unit length [14]. Given d, k and assuming a constant number of runs of zeros, label the runs by integers modulo m and consider block codes over the ring of integers modulo m —the smallest possible m depends on d and k .

Our approach differs from the one in [14] in two ways. First, we relax the unit length runlength of the ones in [14] (which was motivated by magnetic recording applications). Second, we consider lattices rather than codes over the integers modulo m to allow a wider choice of parameters. Indeed our deletion codes are obtained as sets of vectors in a lattice with a given Manhattan norm. By varying this norm, a single lattice, possibly obtained from a single Lee code by Construction A, can produce an infinity of deletion codes. We extend some results of [1], [21] on generalized theta series, called there ν -series, to effectively enumerate these special sets of vectors in the lattice. In particular, if the lattice is obtained via Construction A from a code, the generalized ν -series allows to enumerate these sets from the weight enumerators of the code.

The paper is organized as follows. In Section II, we formalize the problem. In Section III, we determine the sizes of codes derived from Construction A lattices. In Section IV we provide a codebook generation algorithm and a corresponding decoding algorithm for a specific class of lattices which includes the E_8 lattice. In Section V, using tools developed in Section III we derive the analogue of the Gilbert and Hamming bounds for the Manhattan metric space. In Section VI we derive the asymptotic versions of these bounds. In Section VII, we provide a few concluding remarks and point to some open problems.

II. BACKGROUND AND STATEMENT OF THE PROBLEM

Consider a binary sequence of length N that starts with a zero and that contains an even number n of runs—hence $n/2$ runs of zeros and $n/2$ runs of ones. For instance, the sequence 0011100011 corresponds to $N = 10$ and $n = 4$. Throughout the paper we make the following hypothesis:

Working hypothesis. *In any given code n is the same across codewords and they all start with a zero. Moreover, the runlengths in each codeword are supposed to be lower bounded by some constant $r \geq 1$ where $r - 1$ corresponds to the maximum number of deletions that can occur over a length N codeword. This condition is imposed so that the number of runs before and after transmission remains the same.*

With a given length N binary sequence we associate its corresponding runlength sequence

$$(x_1, y_1, \dots, x_i, y_i, \dots, x_{n/2}, y_{n/2})$$

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where x_i and y_i denote the i th runlength of zeros and ones, respectively. For instance, sequence 0011100011 corresponds to $(2, 3, 3, 2)$. The integer sequence so constructed satisfies the constraint

$$N = \sum_{i=1}^{n/2} (x_i + y_i).$$

Denote by ϕ the above correspondence from \mathbb{F}_2^N to \mathbb{Z}^n . The **Levenshtein distance** between two binary vectors is the least number of deletions to go from one to the other [15]. The **Manhattan distance** between two vectors $\mathbf{w}, \mathbf{z} \in \mathbb{Z}^n$ is defined as

$$|\mathbf{w} - \mathbf{z}| \stackrel{\text{def}}{=} \sum_{i=1}^n |w_i - z_i|.$$

The following observation is trivial but crucial.

Proposition 1. *Under the above working hypothesis, the map ϕ is an isometry between \mathbb{F}_2^N with the Levenshtein distance and \mathbb{Z}^n with the Manhattan distance.*

Proof: Let

$$\mathbf{z} = (x_1, y_1, \dots, x_n, y_n)$$

denote a sequence of runs. Let j be an integer $\leq r - 1$. Any deletion of j zeros (resp. ones) into run number i will result into a change of x_i (resp. y_i) into $x_i \pm j$ (resp. $y_i \pm j$) yielding a sequence \mathbf{z}' at Manhattan distance j away from \mathbf{z} . ■

The problem we consider is to characterize $A(n, d, N, r)$, the largest number of length n vectors of nonnegative integers at Manhattan distance at least d apart and with coordinates summing up to N . Any set of length n vectors with integral entries $\geq r$, at Manhattan distance at least d apart, and coordinates summing up to N , we refer to as an (n, d, N, r) -set.

III. ENUMERATION FOR CONSTRUCTION A LATTICES

A **code** $C \subseteq \mathbb{Z}_m^n$ is defined as a \mathbb{Z}_m -submodule of \mathbb{Z}_m^n . The **complete weight enumerator** (cwe) of C is defined as the polynomial (see [22, Chap. 5.6])

$$\text{cwe}_C(x_1, x_2, \dots, x_m) = \sum_{c \in C} \prod_{i=0}^{m-1} x_i^{n_i(c)},$$

where $n_i(c)$ is the number of entries equal to i in the vector c . For $m = 2$, we let

$$W_C(x, y) \stackrel{\text{def}}{=} \text{cwe}_C(x, y)$$

be the classical **weight enumerator** of a binary code.

A **lattice** of \mathbb{R}^n is defined as a discrete additive subgroup of \mathbb{R}^n . A lattice L is said to be obtained by **Construction A** from a code C of \mathbb{Z}_m^n if C is the image of L by reduction modulo m componentwise [8, Chap. 7.2]. Such a lattice is denoted by $L = A(C)$. An important parameter of a lattice is its minimum distance (norm) which is given by the following proposition. Recall that the **Lee weight** of a symbol $x \in \mathbb{Z}_m = \{0, 1, \dots, m-1\}$ is defined as

$$\min(x, m - x).$$

The weight of a vector is the sum of the weights of its components, and the Lee distance of two vectors is the Lee weight of their difference vector. The **Lee distance** of a linear code $C \subseteq \mathbb{Z}_m^n$ is the minimum weight of its nonzero elements.

Proposition 2 ([19]). *Let $L = A(C)$ for some $C \subseteq \mathbb{Z}_m^n$. Then the minimum distance of L is given by*

$$d = \min(d', m)$$

where d' is the minimum Lee distance of C .

For an integer $r \geq 0$ define

$$\nu_L(r; q) \stackrel{\text{def}}{=} \sum_{\substack{\mathbf{x} \in L: \\ \min_i x_i \geq r}} q^{|\mathbf{x}|}$$

as the shifted ν -series in the indeterminate q of the lattice L .

This definition extends trivially to any discrete subset L of \mathbb{R}^n . The motivation for this generating function, whose case $r = 0$ is the ν -series of [1], [20], stems from Proposition 3 below which gives a lower bound on $A(n, d, N, r)$.

Notation. We use the Waterloo notation for coefficients of generating series (see [13]). Given q -series $f = \sum_i f_i q^i$ we denote by $[q^i]f(q)$ the coefficient f_i .

Proposition 3. *If L is a lattice of \mathbb{R}^n with minimum Manhattan distance d then the set of vectors of L with coordinate entries bounded below by r and Manhattan norm N forms an (n, d, N, r) -set of size $[q^N]\nu_L(r; q) \leq A(n, d, N, r)$.*

The proof of Proposition 3 immediately follows from the definition of $[q^N]\nu_L(r; q)$ and $A(n, d, N, r)$.

We now show how to compute (shifted) ν -series of lattices from (complete) weight enumerators of codes.

Theorem 1. *If $L = A(C)$ and $m = 2$ then*

$$\nu_L(r; q) = W_C\left(\frac{q^a}{1 - q^2}, \frac{q^b}{1 - q^2}\right),$$

where a (resp. b) is the first even (resp. odd) integer $\geq r$. If $L = A(C)$ and $m = 4$, then

$$\nu_L(r; q) = cwe_C\left(\frac{q^a}{1 - q^4}, \frac{q^b}{1 - q^4}, \frac{q^c}{1 - q^4}, \frac{q^d}{1 - q^4}\right),$$

where a, b, c, d are the first integers $\geq r$, congruent to $0, 1, 2, 3$ modulo 4 respectively.

Proof: Use the same argument as in [1], [21] and write $A(C)$ as a disjoint union of cosets of $m\mathbb{Z}^n$

$$\nu_L(r; q) = W_C(\nu_{2\mathbb{Z}}(r; q), \nu_{2\mathbb{Z}+1}(r; q))$$

for $m = 2$, and

$$\nu_L(r; q) = cwe_C(\nu_{4\mathbb{Z}}(r; q), \nu_{4\mathbb{Z}+1}(r; q), \nu_{4\mathbb{Z}+2}(r; q), \nu_{4\mathbb{Z}+3}(r; q))$$

for $m = 4$, respectively. The result follows by observing that

$$\nu_{4\mathbb{Z}}(r; q) = \frac{q^a}{1 - q^4}$$

and by summing the appropriate geometric series of reason q^2 or q^4 . ■

In Column 2 of Tables I, II, and III, we list for some values of N and r the lower bound $[q^N]\nu_L(r; q)$ to $A(n, d, N, r)$ for the well-known lattices E_8 , BW_{16} , and Λ_{24} . These lattices are constructed from the extended Hamming code H_8 modulo 2 or the Klemm code K_8 modulo 4 for E_8 , the code $RM(1, 4) + 2RM(2, 4)$ for BW_{16} , and the lifted Golay code $\mathcal{Q}R_{24}$ for Λ_{24} . Here $K_s = R_s + 2P_s$ where R_s denotes the length- s repetition code, where $P_s = R_s^\perp$ denotes its dual code, and where $RM(k, m)$ denotes the order- k Reed-Muller code of length 2^m .

Some cwe's for these codes can be found in [2], [3] while others were computed using Magma [4]. The cwe of K_n is easily seen to be

$$\frac{1}{2}[(x_0 + x_2)^n + (x_0 - x_2)^n + (x_1 + x_3)^n + (x_1 - x_3)^n].$$

These numerical results show, for instance, that for $r = 2$ and $N = 64$, among the three lattices E_8 , BW_{16} and Λ_{24} , BW_{16} achieves the best lower bound while Λ_{24} achieves the best bound for $r = 1$ and $N = 64$.

We now add an extra ingredient to the above construction which improves the lower bound on $A(n, d, N, r)$ for N large enough. Let L be a Construction A lattice in \mathbb{Z}^{n-1} with L^1 -distance d . From this lattice in \mathbb{Z}^{n-1} we construct a new set of points in \mathbb{Z}^n as

$$\hat{L} \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_{n-1}, N - \sum_{i=1}^{n-1} x_i) \mid (x_1, \dots, x_{n-1}) \in L\}.$$

Note that the map

$$(x_1, x_2, \dots, x_{n-1}) \mapsto (x_1, x_2, \dots, x_{n-1}, N - \sum_{i=1}^{n-1} x_i)$$

is the Manhattan analogue map of the Yaglom map (see, e.g., [8, Chap. 9, Theorem 6])

$$(x_1, x_2, \dots, x_{n-1}) \mapsto (x_1, x_2, \dots, x_{n-1}, (N^2 - \sum_{i=1}^{n-1} x_i^2)^{1/2})$$

from \mathbb{R}^{n-1} to \mathbb{R}^n .

Column 3 of Tables I and II gives the lower bound $[q^N]\nu_{\hat{L}}(r; q)$ for the secondly proposed code construction. As we can observe, for N large enough (e.g., $N \geq 28$ for E_8), this second construction improves the first.

In this section we derived lower bounds on $A(n, d, N, r)$ in a non-constructive fashion from the properties of L and \hat{L} using generating functions (Proposition 3). In the next section we provide an explicit code construction for a specific family of lattices along with an effective decoding algorithm.

TABLE I
SIZE $[q^N]\nu_L(r; q)$ OF (n, d, N, r) — SET WITH $L = A(H_8)$, $d \geq 2$ AND $r = 1, 2$

N	$[q^N]\nu_{E_8}(1; q)$	$[q^N]\nu_{\widehat{E}_8}(1; q)$
8	1	0
10	8	1
12	50	9
14	232	59
16	835	291
18	2480	1126
20	6372	3606
22	14640	9978
24	30789	24618
26	60280	55407
28	111254	115687
30	195416	226941
32	329095	422357
34	534496	751452
36	841160	1285948
N	$[q^N]\nu_{E_8}(2; q)$	$[q^N]\nu_{\widehat{E}_8}(2; q)$
16	1	0
18	8	1
20	50	9
22	232	59
24	835	291
26	2480	1126
28	6372	3606
30	14640	9978
32	30789	24618
34	60280	55407
36	111254	115687
38	195416	226941
40	329095	422357
42	534496	751452
44	841160	1285948

IV. CODE CONSTRUCTION AND DECODING ALGORITHM

In this section, we describe two algorithms with respect to the lattice $A(K_n)$:

- a search algorithm that generates explicitly an (n, N, d, r) set carved from the lattice;
- a corresponding decoding algorithm.

Define code

$$C(n, d, N, r) \stackrel{\text{def}}{=} \{\mathbf{c} \in A(K_n) : \min_i c_i \geq r, \sum_{i=1}^n c_i = N\}$$

and note that the minimum distance of $C(n, d, N, r)$ is at least 4, the minimum distance inherited from $A(K_n)$. The generator matrix G for the lattice $A(K_n)$ is

$$G = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 2 \\ 0 & 0 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 4 \end{pmatrix}$$

hence any codeword \mathbf{c} in $C(n, d, N, r)$ can be expressed as

$$\mathbf{c} = (x_1, x_1 + 2x_2, \dots, x_1 + 2x_{n-1}, x_1 + 2 \sum_{i=2}^{n-1} x_i + 4x_n)$$

with

$$l_i \leq x_i \leq u_i$$

and where l_i and u_i are determined as follows.

Define

$$S_i \stackrel{\text{def}}{=} x_1 + \sum_{j=2}^i (x_1 + 2x_j)$$

TABLE II
SIZE $[q^N]\nu_L(r; q)$ OF (n, d, N, r) —SET WITH $L = A(K_8)$, $d \geq 4$ AND $r = 1, 2$

N	$[q^N]\nu_{E_8}(1; q)$	$[q^N]\nu_{\widehat{E}_8}(1; q)$
8	1	0
12	36	1
16	331	37
20	1752	368
24	6765	2120
28	21164	8885
32	56823	30049
36	135728	86872
40	295545	222600
44	596980	518145
48	1133187	1115125
52	2041480	2248312
56	3517605	4289792
60	5832828	7807397
64	9354095	13640225
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N	$[q^N]\nu_{E_8}(2; q)$	$[q^N]\nu_{\widehat{E}_8}(1; q)$
16	1	0
20	36	1
24	331	37
28	1752	368
32	6765	2120
36	21164	8885
40	56823	30049
44	135728	86872
48	295545	222600
52	596980	518145
56	1133187	1115125
60	2041480	2248312
64	3517605	4289792
68	5832828	7807397
72	9354095	13640225

and

$$T \stackrel{\text{def}}{=} x_1 + 2 \sum_{j=2}^{n-1} x_j.$$

Then

- for $i = 1$,

$$\begin{aligned} l_1 &= r \\ u_1 &= N - (n - 1)r, \end{aligned}$$

- for $2 \leq i \leq n - 1$,

$$\begin{aligned} l_i &= \left\lceil \frac{1}{2}(r - x_1) \right\rceil \\ u_i &= \left\lfloor \frac{1}{2}(N - (n - i)r - S_{i-1} - x_1) \right\rfloor, \end{aligned}$$

- for $i = n$,

$$\begin{aligned} l_n &= \left\lceil \frac{1}{4}(r - T) \right\rceil \\ u_n &= \left\lfloor \frac{1}{4}(N - (n - 1)r - T) \right\rfloor. \end{aligned}$$

Searching the codewords can be done by a tree search through all nodes from level 1 (corresponding to x_1) to level n (corresponding to x_n). With the above constraints, we are able to efficiently generate all codewords in $C(n, d, N, r)$. Numerical results are given in Table IV.

Table V gives for $n = 8$, $N = 12$, $r = 1$ and the quaternary lattice $E_8 = A(K_8)$ the number of visited nodes at level i and its naive upper bound which is roughly $(N - 7)(\frac{N-6}{2})^{i-1}$, for different i 's. Table VI gives the number of visited nodes at level $i = 6$ for different values of N (we keep $n = 8$ and $r = 1$).

We now turn to decoding. Recall that in [6] the decoding of a Construction A q -ary lattice for the L^1 -norm is reduced to that of a q -ary linear code for the Lee metric.

We now describe our decoding algorithm for the $C(n, N, d, r)$ code (carved from $A(K_n)$) using the runlength limited (RLL) sequence of its codewords. Recall that, because of our working hypothesis, the channel preserves the number of runs.

From the definition of $A(K_n)$ we have

$$A(K_n) = 2D_n \cup (\mathbf{1} + 2D_n),$$

TABLE III
SIZE $[q^N]\nu_L(r; q)$ OF (n, d, N, r) -SET WITH $L = BW_{16}, \Lambda_{24}, d \geq 4$ AND $r = 1, 2$

N	$[q^N]\nu_{BW_{16}}(1; q)$	N	$[q^N]\nu_{\Lambda_{24}}(1; q)$
16	1	24	1
20	16	28	24
24	306	32	300
28	3984	36	2600
32	39235	40	23415
36	310176	44	299760
40	2016996	48	4144211
44	11005344	52	48058824
48	51463749	56	448956690
52	210557360	60	3450990152
56	767796630	64	22448210613
60	2535136560	68	126639274800
64	7680579975	72	632120648146
68	21588192576	76	2837407970784
72	56814408136	80	11605964888130
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N	$[q^N]\nu_{BW_{16}}(2; q)$	N	$[q^N]\nu_{\Lambda_{24}}(2; q)$
32	1	48	1
36	16	52	24
40	306	56	300
44	3984	60	2600
48	39235	64	23415
52	310176	68	299760
56	2016996	72	4144211
60	11005344	76	48058824
64	51463749	80	448956690
68	210557360	84	3450990152
72	767796630	88	22448210613
76	2535136560	92	126639274800
80	7680579975	96	632120648146
84	21588192576	100	2837407970784
88	56814408136	104	11605964888130

TABLE IV
CODEWORDS IN E_8 WITH RLL REPRESENTATION FOR $r = 1, N = 12$

(5, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1, 3, 1, 1, 1, 3)
(3, 1, 1, 1, 1, 1, 1, 3)	(1, 1, 1, 3, 1, 1, 3, 1)
(3, 1, 1, 1, 1, 1, 3, 1)	(1, 1, 1, 3, 1, 3, 1, 1)
(3, 1, 1, 1, 1, 3, 1, 1)	(1, 1, 1, 3, 3, 1, 1, 1)
(3, 1, 1, 1, 3, 1, 1, 1)	(1, 1, 1, 5, 1, 1, 1, 1)
(3, 1, 1, 3, 1, 1, 1, 1)	(1, 1, 3, 1, 1, 1, 1, 3)
(3, 1, 3, 1, 1, 1, 1, 1)	(1, 1, 3, 1, 1, 1, 3, 1)
(3, 3, 1, 1, 1, 1, 1, 1)	(1, 1, 3, 1, 1, 3, 1, 1)
(1, 1, 1, 1, 1, 1, 1, 5)	(1, 1, 3, 1, 3, 1, 1, 1)
(1, 1, 1, 1, 1, 1, 3, 3)	(1, 1, 3, 3, 1, 1, 1, 1)
(1, 1, 1, 1, 1, 1, 5, 1)	(1, 1, 5, 1, 1, 1, 1, 1)
(1, 1, 1, 1, 1, 3, 1, 3)	(1, 3, 1, 1, 1, 1, 1, 3)
(1, 1, 1, 1, 1, 3, 3, 1)	(1, 3, 1, 1, 1, 1, 3, 1)
(1, 1, 1, 1, 1, 5, 1, 1)	(1, 3, 1, 1, 1, 3, 1, 1)
(1, 1, 1, 1, 3, 1, 1, 3)	(1, 3, 1, 1, 3, 1, 1, 1)
(1, 1, 1, 1, 3, 1, 3, 1)	(1, 3, 1, 3, 1, 1, 1, 1)
(1, 1, 1, 1, 3, 3, 1, 1)	(1, 3, 3, 1, 1, 1, 1, 1)
(1, 1, 1, 1, 5, 1, 1, 1)	(1, 5, 1, 1, 1, 1, 1, 1)

where

$$D_n \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i \equiv 0 \pmod{2}\}.$$

It is clear that D_n contains

$$A_{n-1} = \{\mathbf{x} \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i = 0\}$$

as a sublattice.

Following [7], we reduce the decoding in $2D_n$ to the decoding in $2A_{n-1}$ by noting that

$$2D_n = k + 2A_{n-1}$$

TABLE V
NUMBER OF VISITED NODES AND ITS UPPER BOUND OF SEARCHING CODEWORDS FROM $E_8 = A(K_8)$ WITH $r = 1, N = 12$

Level	#nodes	Upper bound
2	9	15
3	11	45
4	16	135
5	21	405
6	28	1215
7	36	3645

TABLE VI
NUMBER OF VISITED NODES AND ITS UPPER BOUND OF SEARCHING CODEWORDS FROM $E_8 = A(K_8)$ FOR $r = 1$

N	#nodes(level 7)	#nodes(level 6)	Upper bound(level 6)
8	1	1	1
12	36	28	1215
16	331	217	28125
20	1752	1008	218491
24	6765	3465	1003833
28	21164	9724	3382071
32	56823	23569	9282325
36	135728	51136	22021875
40	295545	101745	46855281
44	596980	188860	91615663
48	1133187	331177	167448141
52	2041480	553840	289635435
56	3517605	889785	478515625
60	5832828	1381212	760492071
64	9354095	2081185	1169135493

with $k = (N, 0, \dots, 0)$.

The following lemma allows us to find a closest codeword in A_{n-1} to a received vector in \mathbb{Z}^n .

Lemma 1. Any vector of coordinates summing up to s in \mathbb{Z}_+^n is at L^1 -distance at least $|s|$ from any vector in A_{n-1} .

Proof: Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n$ with $\sum_{i=1}^n x_i = s$ and $\mathbf{y} \in A_{n-1}$. Then

$$|\mathbf{x} - \mathbf{y}| = \sum_{i=1}^n |x_i - y_i| \geq \left| \sum_{i=1}^n (x_i - y_i) \right| = s$$

since $\sum_{i=1}^n y_i = 0$. ■

Proposition 4. Let

$$\phi^{(i)} : \mathbb{Z}^n \rightarrow A_{n-1} \tag{1}$$

$$\mathbf{x} \mapsto (\phi_1^{(i)}, \phi_2^{(i)}, \dots, \phi_n^{(i)}), \tag{2}$$

where

$$\phi_j^{(i)} = \begin{cases} x_j - (x_1 + \dots + x_n) & \text{if } j = i \\ x_j & \text{if } j \neq i. \end{cases}$$

Then for any $\mathbf{x} \in \mathbb{Z}_+^n$, $\phi^{(i)}(\mathbf{x})$ is a closest point of A_{n-1} to \mathbf{x} .

Proof: The proof follows from Lemma 1 with $s = |\mathbf{x}|$. ■

In case of a single deletion error (recall that the minimum distance of $A(K_n)$ is 4), there exists a unique $i \in \{1, 2, \dots, n\}$ such that $2A_{n-1}$ contains $\phi^{(i)}(\mathbf{x})$. That i is where the error occurs.

Algorithm

Input: A received vector \mathbf{x} of length n

Output: A nearest codeword $\hat{\mathbf{x}}$ to \mathbf{x}

- 1) $N \leftarrow$ length of the binary code corresponding
- 2) $\mathbf{a} \leftarrow$ a coset representative of A_{n-1} in D_n
- 3) if $\sum_{i=1}^n \mathbf{x}[i] == N - 1$ then
- 4) $\hat{\mathbf{x}} \leftarrow \mathbf{x}$
- 5) Find (the unique) coordinate $\hat{\mathbf{x}}[j]$ whose parity is different from the others

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6)  $\hat{\mathbf{x}}[j] \leftarrow \hat{\mathbf{x}}[j] + 1$ 
7) else
8)  $\hat{\mathbf{X}} \leftarrow \mathbf{x} - \mathbf{a}$ 
9)  $s \leftarrow \sum_{i=1}^n \hat{X}[i]$ 
10) for  $i \leftarrow 1$  to  $n$  do
11)  $\hat{\mathbf{x}} \leftarrow \hat{\mathbf{X}}$ 
12)  $\hat{\mathbf{x}}[i] \leftarrow \hat{\mathbf{x}}[i] - s$ 
13) if all coordinates of  $\hat{\mathbf{x}}$  are even then
14) break
15) end if
16) end for
17)  $\hat{\mathbf{x}} \leftarrow \hat{\mathbf{x}} + \mathbf{a}$ 
18) end if
19) return  $\hat{\mathbf{x}}$ 

```

The complexity of our algorithm can be calculated as follows:

- line 3 requires $n - 1$ additions
- line 8 requires n additions
- line 9 requires $n - 1$ additions
- lines 10 to 16 require one addition (plus one parity test) for n times
- line 17 requires n additions

Thus the decoding algorithm requires $5n - 2$ additions over \mathbb{Z} plus n parity tests.

For instance, take $n = 8, N = 12, r = 1$ and consider $\mathbf{x} = (3, 2, 1, 1, 1, 1, 1, 1)$ as a received word. The code $C(8, 12, 1)$ has 36 codewords and has minimum distance 4. By taking as coset representative of A_{n-1} in D_n

$$\mathbf{a} = (1, 1, 1, 1, 1, 1, 1, 5),$$

the nearest codewords in A_{n-1} to $\mathbf{x} - \mathbf{a}$ are

$$\begin{aligned}
\phi^{(1)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 0, 0, 0, 0, 0, -4), \\
\phi^{(2)}(\mathbf{x} - \mathbf{a}) &= (2, 2, 0, 0, 0, 0, 0, -4), \\
\phi^{(3)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 1, 0, 0, 0, 0, -4), \\
\phi^{(4)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 0, 1, 0, 0, 0, -4), \\
\phi^{(5)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 0, 0, 1, 0, 0, -4), \\
\phi^{(6)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 0, 0, 0, 1, 0, -4), \\
\phi^{(7)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 0, 0, 0, 0, 1, -4), \\
\phi^{(8)}(\mathbf{x} - \mathbf{a}) &= (2, 1, 0, 0, 0, 0, 0, -3).
\end{aligned}$$

Since $\phi^{(2)}(\mathbf{x} - \mathbf{a})$ is the only codeword in $2A_{n-1}$, we decode $\mathbf{x} = (3, 2, 1, 1, 1, 1, 1, 1)$ since

$$\phi^{(2)}(\mathbf{x} - \mathbf{a}) + \mathbf{a} = (3, 3, 1, 1, 1, 1, 1, 1).$$

V. BOUNDS ON $A(n, d, N, r)$

First we recall a well-known identity of formal power series.

Lemma 2. For any integer $n \geq 1$, we have

$$\frac{1}{(1-q)^n} = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} q^i.$$

Proof: Differentiate the geometric series

$$\frac{1}{(1-q)} = \sum_{i=0}^{\infty} q^i$$

with respect to q and use induction on n . ■

Using generating functions, we compute the volume $V(n, e)$ of the Manhattan ball of radius e in \mathbb{Z}^n .

Lemma 3. For any integers $n \geq e \geq 1$, we have

$$V(n, e) = [q^e] \frac{(1+q)^n}{(1-q)^{n+1}} = \sum_{i=0}^{\min(n,e)} 2^i \binom{n}{i} \binom{e}{i}.$$

Proof:

$$\begin{aligned} V(n, e) &= \sum_{i=0}^e [q^i] \nu_{\mathbb{Z}^n}(-\infty, q) \\ &= \sum_{i=0}^e [q^i] \left(\frac{1+q}{1-q} \right)^n \\ &= [q^e] \frac{(1+q)^n}{(1-q)^{n+1}}. \end{aligned}$$

The second expression in the Lemma is from [10]. It can be rederived from the above generating series by expanding

$$\left(1 + \frac{2q}{1-q}\right)^{n+1} = \sum_{i=0}^n \binom{n}{i} 2^i \frac{q^i}{(1-q)^{i+1}}$$

through Lemma 2. ■

By the same techniques, we can compute the volume of the ambient space $A(n, 1, N, r)$.

Lemma 4. For any integer $N > nr$ and $r > e \geq 1$, we have

$$A(n, 1, N, r) = \binom{N - nr + n - 1}{n - 1}.$$

Proof:

$$\begin{aligned} A(n, 1, N, r) &= [q^N] \nu_{\mathbb{Z}^n}(r, q) = [q^N] \left(q^r \frac{1}{1-q} \right)^n \\ &= [q^{N-nr}] \frac{1}{(1-q)^n}. \end{aligned}$$

The result follows from Lemma 2. ■

We are now in a position to formulate the analogues of the Gilbert and Hamming bound in the present context.

Theorem 2. For any integers $N > nr$, $n \geq d$, and $r > e = \lfloor (d-1)/2 \rfloor \geq 1$, we have

$$\frac{\binom{N-nr+n-1}{n-1}}{V(n, d-1)} \leq A(n, d, N, r) \leq \frac{\binom{N-nr+n-1}{n-1}}{V(n, e)}.$$

Proof: Combine Lemma 3 and Lemma 4 with the standard arguments. ■

The lower and upper bounds on $A(n, d, N, r)$ in Theorem 2 are given in Table VII and Table VIII for lattices E_8 and BW_{16} . In these tables we defined

$$I(n, d, N, r) \stackrel{\text{def}}{=} \left\lceil \frac{\binom{N-nr+n-1}{n-1}}{V(n, d-1)} \right\rceil$$

and

$$S(n, e, N, r) \stackrel{\text{def}}{=} \left\lceil \frac{\binom{N-nr+n-1}{n-1}}{V(n, d-1)} \right\rceil.$$

The numerical results show that $[q^N] \nu_L(r; q)$ (a lower bound to $A(n, d, N, r)$ by Proposition 3), lies between $I(n, d, N, r)$ and $S(n, e, N, r)$ for many parameter values. Exceptions are, for instance, for BW_{16} with $r = 2$, and $N = 48, \dots, 96$. Whether these code constructions yield sizes between $I(n, d, N, r)$ and $S(n, e, N, r)$ for large N is an open issue.

Since all codewords have constant Manhattan distance, it is natural to consider the Johnson bound in the Lee metric:

Theorem 3. If $d > N(1 - 1/2n)$, then we have

$$A(n, d, N, r) \leq \frac{d}{d - N(1 - 1/2n)}.$$

Proof: Reduce all vectors modulo $Q = 2N$. Use Lemma 13.62 of [5] with $\overline{D} = Q/4 = N/2$, and $x = 1/n$. ■

TABLE VII
BOUNDS ON $A(n, d, N, r)$ WITH $L = E_8$ AND $r = 2, 3, 4$

N	$I(8, 4, N, 2)$	$[q^N] \nu_{E_8}(2; q)$	$S(8, 1, N, 2)$
24	8	331	378
28	61	1752	2964
32	295	6765	14421
36	1067	21164	52237
40	3157	56823	154680
44	8073	135728	395560
48	18465	295545	904761
52	38685	596980	1895536
56	75500	1133187	3699499
60	138986	2041480	6810300
64	243611	3517605	11936925
68	409544	5832828	20067614
72	664191	9354095	32545333
76	1043996	14567520	51155776
80	1596508	22105457	78228865
N	$I(8, 4, N, 3)$	$[q^N] \nu_{E_8}(3; q)$	$S(8, 1, N, 3)$
32	8	331	378
36	61	1752	2964
40	295	6765	14421
44	1067	21164	52237
48	3157	56823	154680
52	8073	135728	395560
56	18465	295545	904761
60	38685	596980	1895536
64	75500	1133187	3699499
68	138986	2041480	6810300
72	243611	3517605	11936925
76	409544	5832828	20067614
80	664191	9354095	32545333
84	1043996	14567520	51155776
88	1596508	22105457	78228865
N	$I(8, 4, N, 4)$	$[q^N] \nu_{E_8}(4; q)$	$S(8, 1, N, 4)$
40	8	331	378
44	61	1752	2964
48	295	6765	14421
52	1067	21164	52237
56	3157	56823	154680
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64	18465	295545	904761
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72	75500	1133187	3699499
76	138986	2041480	6810300
80	243611	3517605	11936925
84	409544	5832828	20067614
88	664191	9354095	32545333
92	1043996	14567520	51155776
96	1596508	22105457	78228865

VI. ASYMPTOTIC BOUNDS ON $A(n, d, N, r)$

We assume that r is fixed, that $N \rightarrow \infty$, and that $n \sim \eta N/r$, $d \sim \delta N$ for some constants η, δ with $\eta \in (0, 1)$, and $\delta \geq 0$. Because each codeword has weight N , the triangle inequality in the Manhattan metric shows that $\delta \in (0, 2)$. Denote by R the asymptotic exponent of $A(n, d, N, r)$, that is

$$R \stackrel{\text{def}}{=} \limsup \frac{1}{N} \log_2 A(n, d, N, r).$$

The asymptotic form of Theorem 3 shows that $\delta \in (0, 1)$ whenever $R \neq 0$.

Let

$$L(x) = x \log_2 x + \log_2(x + \sqrt{x^2 + 1}) - x \log_2(\sqrt{x^2 + 1} - 1).$$

It was proved in [9] that when $x \rightarrow \infty$ and $e \sim \epsilon n$

$$\lim \frac{1}{n} \log_2 V(n, e) = L(\epsilon).$$

For convenience, let

$$H(q) \stackrel{\text{def}}{=} -q \log_2 q - (1 - q) \log_2(1 - q)$$

denote the binary entropy function and let

$$f(x, y, z) \stackrel{\text{def}}{=} [1 - y + y/x] H\left(\frac{y}{y + x(1 - y)}\right) - (y/x) L\left(\frac{xz}{y}\right).$$

We establish the asymptotic version of Theorem 2.

TABLE VIII
BOUNDS ON $A(n, d, N, r)$ WITH $L = BW_{16}$ AND $r = 2, 3, 4$

N	$I(16, 4, N, 2)$	$[q^N] \nu_{BW_{16}}(2; q)$	$S(16, 1, N, 2)$
36	1	16	117
40	82	306	14858
44	2890	3984	526783
48	49949	39235	9107278
52	539795	310176	98422520
56	4178302	2016996	761843656
60	25184088	11005344	4591898687
64	124915457	51463749	22776251653
68	529944363	210557360	96626522164
72	1977679995	767796630	360596985630
76	6630474804	2535136560	1208956572561
80	20297778673	7680579975	3700961644542
84	57467324395	21588192576	10478208814512
88	152025004051	56814408136	27719225738485
92	378928483749	141077361984	69091293536850
96	896068510238	332674600329	163383158366718
<hr/>			
N	$I(16, 4, N, 3)$	$[q^N] \nu_{BW_{16}}(3; q)$	$S(16, 1, N, 3)$
52	1	16	117
56	82	306	14858
60	2890	3984	526783
64	49949	39235	9107278
68	539795	310176	98422520
72	4178302	2016996	761843656
76	25184088	11005344	4591898687
80	124915457	51463749	22776251653
84	529944363	210557360	96626522164
88	1977679995	767796630	360596985630
92	6630474804	2535136560	1208956572561
96	20297778673	7680579975	3700961644542
100	57467324395	21588192576	10478208814512
104	152025004051	56814408136	27719225738485
108	378928483749	141077361984	69091293536850
112	896068510238	332674600329	163383158366718
<hr/>			
N	$I(16, 4, N, 4)$	$[q^N] \nu_{BW_{16}}(4; q)$	$S(16, 1, N, 4)$
68	1	16	117
72	82	306	14858
76	2890	3984	526783
80	49949	39235	9107278
84	539795	310176	98422520
88	4178302	2016996	761843656
92	25184088	11005344	4591898687
96	124915457	51463749	22776251653
100	529944363	210557360	96626522164
104	1977679995	767796630	360596985630
108	6630474804	2535136560	1208956572561
112	20297778673	7680579975	3700961644542
116	57467324395	21588192576	10478208814512
120	152025004051	56814408136	27719225738485
124	378928483749	141077361984	69091293536850
128	896068510238	332674600329	163383158366718

Theorem 4. *With the above notation we have*

$$f(r, \eta, \delta) \leq R \leq f(r, \eta, \delta/2).$$

Proof: The result follows from Theorem 2 by standard entropic estimates for binomial coefficients for the numerator and the result on large alphabet Lee balls from [9] for the denominators. ■

In Fig. 1 and 2, the graphs of the asymptotic lower bound curve $f(r, \eta, \delta)$ with different parameters η and $r = 2$ show that the rate R is higher when η is around 0.5.

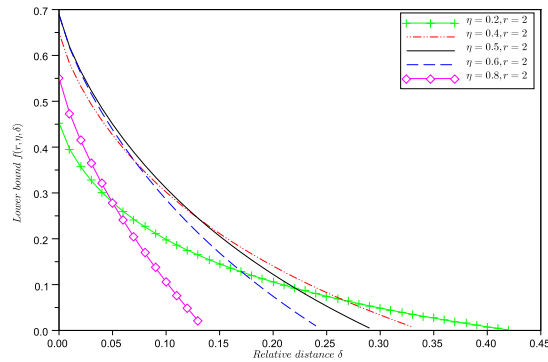


Fig. 1. Graphs of $f(r, \eta, \delta)$ for $r = 2$ and $\eta = 0.2, 0.4, 0.5, 0.6, 0.8$

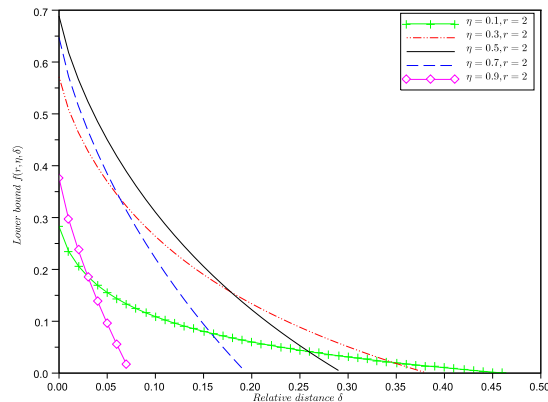


Fig. 2. Graphs of $f(r, \eta, \delta)$ for $r = 2$ and $\eta = 0.1, 0.3, 0.5, 0.7, 0.9$

VII. CONCLUSION AND OPEN PROBLEMS

We approached a problem of binary coding for the Levenshtein distance by using lattices for the Manhattan metric. These lattices are obtained by Construction A applied to binary and quaternary codes. Since decoding these lattices for the Manhattan metric can be reduced to decoding the constructing code for the Lee distance [6], it is worth to investigate the decoding of \mathbb{Z}_4 -codes beyond the Klemm's code considered here. Another approach would be to consider \mathbb{Z}_4 -codes with a known decoding algorithm (e.g., Preparata [11], Goethals [12], Calderbank-MacGuire [18]) and look at the performance of the corresponding lattices.

More generally, it is worth considering larger alphabets like $\mathbb{Z}_8, \mathbb{Z}_{16}$, when building lattices in higher dimensions. The Lee decoding problem for such codes is completely open. Moving away from Construction A, finding the densest lattice for the Manhattan metric in a given dimension is still a deep and fundamental open problem.

Finally, turning to the deletion channel, what allowed us to use algebraic coding techniques was our working hypothesis; the runlengths of each codeword is larger than r , the maximum number of deletions that can occur over the transmission period. Extending these techniques to the case where the working hypothesis does not necessarily hold is an important and challenging open problem.

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